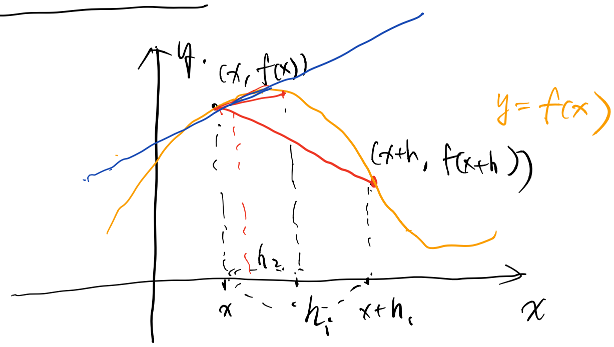
Jan 19

Difference Quotient: Given $f(x)$.



$$\frac{f(x+h)-f(x)}{(x+h)-(x)} = \frac{f(x+h)-f(x)}{h} \leftarrow$$ difference quotient of f (at x, with interval length h).

= slope of the red line

a secant line to the graph of $f(x)$ on the x-y plane

as $h \to 0$, the secant line thru the points $(x, f(x))$ and $(x+h, f(x+h))$ gets closer and closer to the tangent line to the graph of f at the point $(x, f(x))$

so we can use difference quotients for small h to compute the slope of this tangent line

the "limit" of the difference quotient as $h \to 0$

= the slope of the tangent line

=: "the derivative of f at x"

Definition: An "empty set" is a set that has no elements usually denoted by $\{\ \} = \phi$

Ex: A = the set consisting of all students enrolled in this course.

Define B = the set of students from outerspace
$= \phi$

$A \cup \phi = A$ for any set A.

$A \setminus \phi = A$

$\phi \subset A$ for any A.

<u>Subsets</u>: $A \subset B$ if all elements in A are also in B

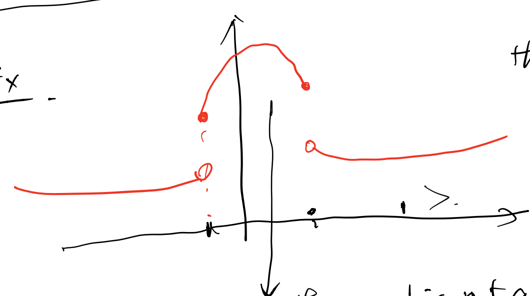
so $B \subset B$

but B is not a "proper subset" of B

$A \subsetneq B$ if $A \subset B$ but $A \neq B$: A is a "proper subset" of B

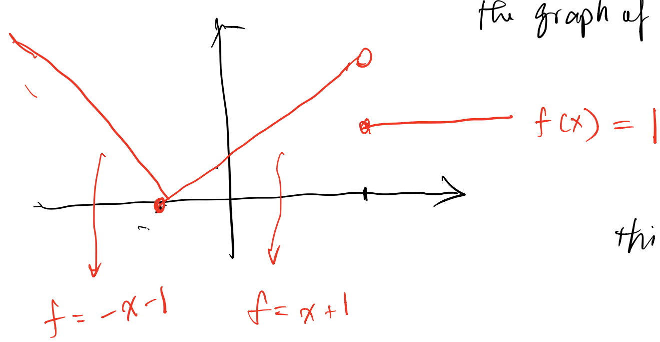
$\phi \subset \phi$ but $\phi \not\subsetneq \phi$.

<u>Ex</u>.



this is a piecewise function, but not piecewise linear.

the graph is not a line, so can not be the graph of a linear function



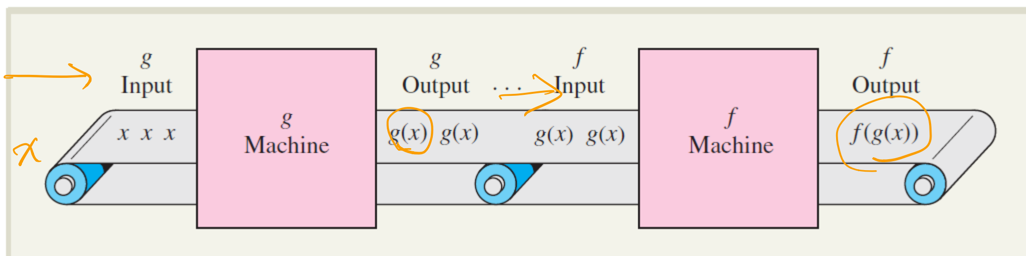
this is a piecewise linear function, which is a piecewise function which is given by a linear function on each interval.

1.5 Composition of functions

Definition 1.5.1. Given functions $f(u)$ and $g(x)$, the composition of f and g, denoted by $(f \circ g)(x)$, is a function of x formed by substituting $u = g(x)$ for u in the formula of $f(u)$, i.e.

$$(f \circ g)(x) = f(g(x)).$$

In the following figure, the definition of composite function is illustrated as an assembly line in which raw input x is first converted into a transitional product $g(x)$ that acts as input in f machine uses to produce $f(g(x))$.



relace x by f(x) to get $g \circ f(x)$

Example 1.5.1. $f(x) = x^2 + 3x + 1$ and $g(x) = x + 1$.
Then

replace x by g(x) = x+1

$$\begin{aligned}(f \circ g)(x) = f(g(x)) &= (g(x))^2 + 3(g(x)) + 1 = (x+1)^2 + 3(x+1) + 1 \\ &= (x^2 + 2x + 1) + (3x + 3) + 1 = x^2 + 5x + 5\end{aligned}$$

Notice that the order of composition matters! unlike products $f \cdot g = g \cdot f$

Similarly,

$$(g \circ f)(x) = g(f(x)) = f(x) + 1 = x^2 + 3x + 2.$$

$x^2 + 3x + 1 + 1$

Remark. In general, $(f \circ g)(x) \neq (g \circ f)(x)$.

set this as a new variable

introducing a new notation $u = x - 1$, $x = u + 1$

Example 1.5.2. Suppose $f(x) = x^3 - 1$ and $h(x) = x - 1$, find $g(u)$ such that $f(x) = g(h(x))$.

want to express f as $g \circ h(x)$ for some g.

Solution.

$$f(x) = x^3 - 1 = (x - 1 + 1)^3 - 1 = (x-1)^3 + 3(x-1)^2 + 3(x-1) = g(u),$$

plug in here

where we define

$$g(u) = u^3 + 3u^2 + 3u.$$

■

Alternatively, use change of variable, express the formula in terms of the new variable u. $x = u+1$

set f as a function of u: $f(x) = (u+1)^3 - 1 = u^3 + 3u^2 + 3u + 1 - 1$
$= u^3 + 3u^2 + 3u = g(u)$

$= g(h(x)) = (g \circ h)(x)$

Example 1.5.3. Suppose $f(x) = (x-5)^2 + \dfrac{3}{(x-5)^3}$, find $g(u)$ and $h(x)$ such that $f(x) = g(h(x))$.

express as $f = g \circ h$

Solution. The form of the given function is

$$f(x) = \square^2 + \frac{3}{\square^3},$$

let $u = x-5 = h(x)$ define $g(u)$

then $f(x) = u^2 + \frac{3}{u^3} =: g(u)$

$= g(h(x))$

where each box contains the expression $x-5$. Thus $f(x) = g(h(x))$, where

$$g(u) = u^2 + \frac{3}{u^3} \text{ and } h(x) = x-5.$$

can also change notation, and denote the variable of the function g as x

$g(x) = x^2 + \frac{3}{x^3}$

■

Definition 1.5.2. A difference quotient for a function $f(x)$ is a composition function of the form

$$\frac{f(x+h) - f(x)}{h}$$

where h is a constant.

Difference quotients are used to compute the slope of a tangent line to the graph and define the derivative, a concept of central importance in calculus.

Example 1.5.4. Find the difference quotient of $f(x) = x^2 - 3x$.

Solution.

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{[(x+h)^2 - 3(x+h)] - [x^2 - 3x]}{h} \\
&= \frac{[x^2 + 2xh + h^2 - 3x - 3h] - [x^2 - 3x]}{h} \\
&= \frac{2xh + h^2 - 3h}{h} = 2x + h - 3.
\end{aligned}$$

■

Geometric interpretation: As slopes of secant lines to the graph of f.

$h \to 0 \rightsquigarrow$ tangent lines. Slopes of tangent lines to the graph of $f \rightsquigarrow$ derivatives of f.

1.6 Modeling in Business and Economics

Example 1.6.1. A manufacturer can produce dinning room tables at a cost of \$200 each. The table has been selling for \$300 each, and at that price consumers have been buying 400 tables per month. The manufacturer is planning to raise the price of the table and estimates that for each \$1 increase in the price, 2 fewer tables will be sold each month. What price corresponds to the maximum profit, and what is the maximum profit?

Solution. Let x be the price. that he wants to get.

original number.

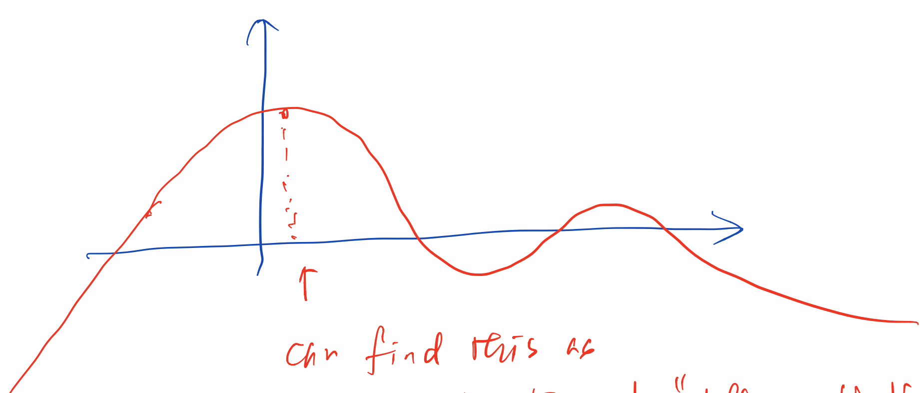
amount of increase in price

$$\begin{aligned}
\text{Profit for one table} &= x - 200 \\
\text{Number of tables sold} &= 400 - 2(x-300) = 1000 - 2x \\
\text{Total profit:} \quad f(x) &= (x-200)(1000-2x) \\
&= -2x^2 + 1400x - 200000 \\
&= -2(x-350)^2 + 45000
\end{aligned}$$

the amount of decrease in number of tables sold.

$f(x)$ is maximized when the manufacturer charges \$350 for each table. ■

Question: How to find max/min for general functions? Calculus helps!



can find this as an application of "differentiation"

MATH1520 University Mathematics for Applications **Spring 2021**

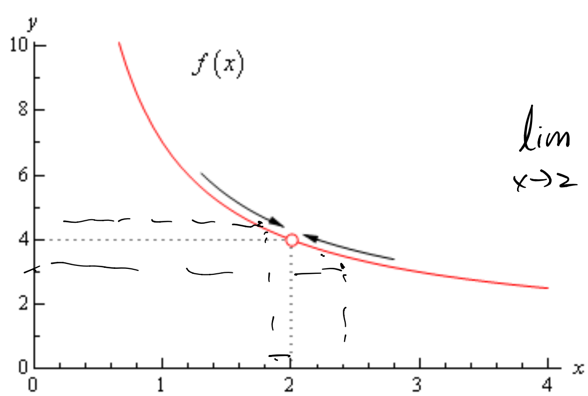
Chapter 2: Limit

Learning Objectives:
(1) Examine the limit concept and general properties of limits.
(2) Compute limits using a variety of techniques.
(3) Compute and use one-sided limits.
(4) Investigate limits involving infinity and "e".

2.1 Limit of a function at one point

(Heuristic) "Definition" 2.1.1. If $f(x)$ gets "closer and closer" to a number L as x gets "closer and closer" to c from *both sides*, then L is called the limit of $f(x)$ as x approaches c, denoted by

$$\lim_{x \to c} f(x) = L.$$



$\lim_{x \to 2} f(x) = 4$

Remark. Limits are defined rigorously via "$\varepsilon - \delta$" language.

Example 2.1.1. Let $f(x) := x + 1$. Find $\lim_{x \to 1} f(x)$

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	2	2.001	2.01	2.1

When x approaches 1 from both sides, $f(x)$ approaches 2. Therefore $\lim_{x \to 1} f(x) = 2$.

$= f(1)$

$\lim_{x \to c} f(x) = f(c)$ only for "good functions" (continuous functions, later)

Remark. 1. The table only gives you an intuitive idea, this is not a rigorous proof.
2. Don't think that the limit is always obtained by substituting $x = 1$ into $f(x)$. The limit only depends on the behavior of $f(x)$ near $x = 1$, but not at $x = 1$.

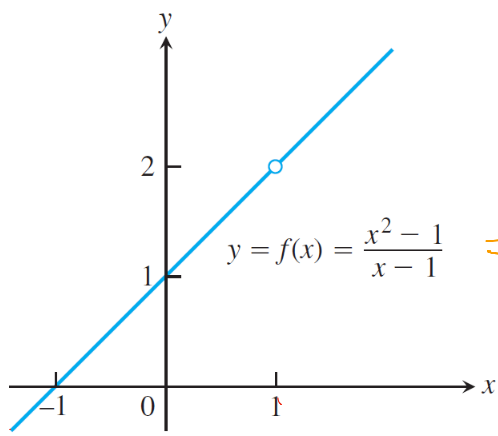
Example 2.1.2. $f(x) = \begin{cases} x+1 & \text{if } x \neq 1, \\ \text{undefined} & \text{if } x = 1. \end{cases}$

x	0.9	0.99	0.999	**1**	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	**undefined**	2.001	2.01	2.1

When x approaches 1 from both sides, $f(x)$ approaches 2. Therefore $\lim_{x\to 1} f(x) = 2$.

limit well-defined even though $f(1)$ is undefined.

Disregard the value of f at 1, the limit of $f(x)$ when x tends to 1 is always 2.



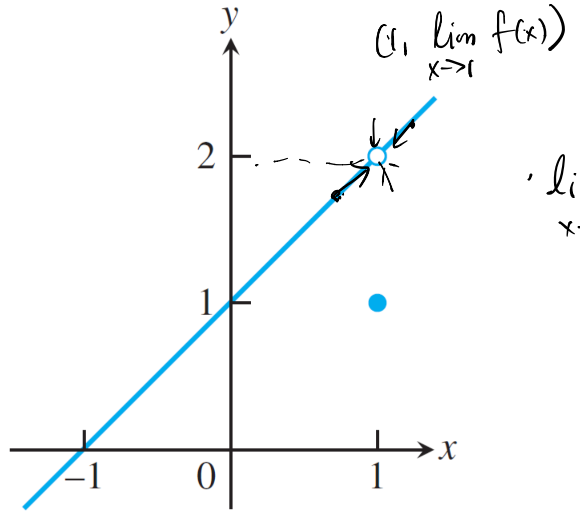
$= \dfrac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} = x+1$ when $x \neq 1$

but f is different from the function $x+1$

Example 2.1.3. $f(x) = \begin{cases} x+1 & \text{if } x \neq 1, \\ 1 & \text{if } x = 1. \end{cases}$

x	0.9	0.99	0.999	**1**	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	**1**	2.001	2.01	2.1

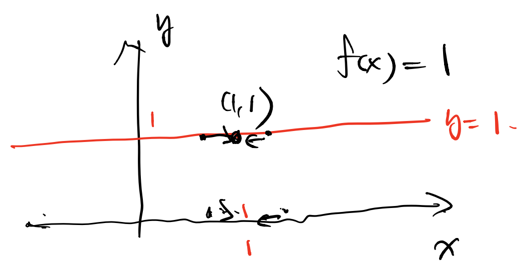
When x approaches 1 from both sides, $f(x)$ approaches 2. Therefore $\lim_{x\to 1} f(x) = 2$.



$\cdot \lim_{x\to 1} f(x) = 2 \neq f(1) = 1$

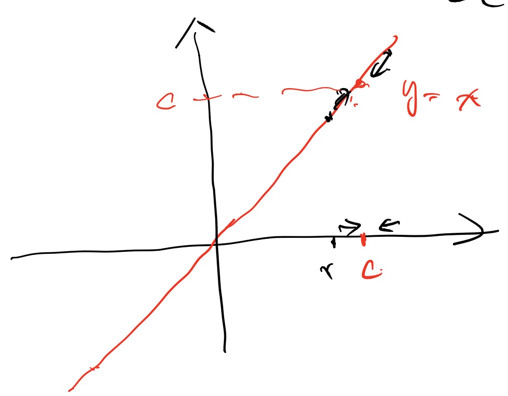
Proposition 1.

1. If $f(x) = k$ is a constant function, then
$$\lim_{x\to c} f(x) = \lim_{x\to c} k = k.$$
For instance, $\lim_{x\to 1} 9 = 9$.



$\lim_{x\to c} f(x) = 1$ for any $c \in \mathbb{R}$

2. If $f(x) = x$, then
$$\lim_{x\to c} f(x) = \lim_{x\to c} x = c.$$
For instance, $\lim_{x\to 3} x = 3$.



Proposition 2. **(Algebraic properties of limits, +, −, ×, ÷)**

If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist (very important!), then

Each formula holds when all terms in the formula is well defined.

1. $\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
2. $\lim_{x\to c}(f(x) - g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$
3. $\lim_{x\to c}(f(x)g(x)) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$

   Especially, $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x)$ for any constant k
4. $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ if $\lim_{x\to c} g(x) \neq 0$.
5. $\lim_{x\to c}(f(x))^p = \left[\lim_{x\to c} f(x)\right]^p$ if $\left[\lim_{x\to c} f(x)\right]^p$ exists

if this exists

Example 2.1.4. Compute the following limits:

1. $\lim\limits_{x\to 1}(x^3+2x-5)$
2. $\lim\limits_{x\to 2}\dfrac{x^4+x^2-1}{x^2+5}$
3. $\lim\limits_{x\to -2}\sqrt{4x^2-3}$

$\lim\limits_{x\to 1} x^3 = (\lim\limits_{x\to 1} x)^3 = 1^3$

Solution.

by rules 1 & 2. (5) (3) $= 2\lim\limits_{x\to 1} x$

1. $\lim\limits_{x\to 1}(x^3+2x-5) = \lim\limits_{x\to 1} x^3 + \lim\limits_{x\to 1} 2x - \lim\limits_{x\to 1} 5 = 1^3 + 2\cdot 1 - 5 = -2.$ Prop1
2. $\lim\limits_{x\to 2}\dfrac{x^4+x^2-1}{x^2+5} = \dfrac{\lim\limits_{x\to 2}(x^4+x^2-1)}{\lim\limits_{x\to 2}(x^2+5)} = \dfrac{\lim\limits_{x\to 2} x^4 + \lim\limits_{x\to 2} x^2 - \lim\limits_{x\to 2} 1}{\lim\limits_{x\to 2} x^2 + \lim\limits_{x\to 2} 5} = \dfrac{19}{9}.$
3. $\lim\limits_{x\to -2}\sqrt{4x^2-3} = \sqrt{\lim\limits_{x\to -2}(4x^2-3)} = \sqrt{\lim\limits_{x\to -2} 4x^2 - \lim\limits_{x\to -2} 3} = \sqrt{16-3} = \sqrt{13}.$

■

Exercise 2.1.1. Compute the following limit:

$$\lim_{x\to 1}\left(x^2 - \frac{3x}{x+5}\right)$$

Example 2.1.5. (Cancelling a common factor)
Find the limit

$$\lim_{x\to 1}\frac{x^2-1}{x^2-3x+2}.$$

Solution. We can't directly use property of division of limit because the denominator $\lim\limits_{x\to 1}(x^2-3x+2) = 1^2 - 3\times 1 + 2 = 0$.

$$\begin{aligned}
\lim_{x\to 1}\frac{x^2-1}{x^2-3x+2} &= \lim_{x\to 1}\frac{(x-1)(x+1)}{(x-1)(x-2)} \\
&= \lim_{x\to 1}\frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(x-2)} \\
&= \lim_{x\to 1}\frac{x+1}{x-2} \\
&= \frac{1+1}{1-2} = -2.
\end{aligned}$$

■